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The $(S)_+$ condition on generalized variational inequalities

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Abstract In this paper, we derive some existence results for generalized variational inequalities associated with mappings satisfying the $(S)_+$ condition. The relation between the $(S)_+$ and $(S)_+^1$ conditions is discussed. As an application, we also consider multivalued complementarity problems associated with mappings satisfying the $(S)_+$ condition, and prove a theorem to characterize the solvability of such problems in terms of exceptional families of elements.

Keywords Generalized variational inequalities · The $(S)_+$condition · Multivalued complementarity problems · Exceptional families of elements

1 Introduction

For any given nonempty sets Ω and W, a mapping $T : \Omega \longrightarrow 2^W$ will be called a multivalued mapping from Ω into W, where 2^W denotes the set of all subsets of W. The set $\mathcal{G}(T) = \{(x, y) \in \Omega \times W : y \in T(x)\}$ is the graph of T.

Throughout this paper, all topological vector spaces are real and Hausdorff. For a topological vector space X, the set of all continuous linear mappings from X into $\mathbb{R}$ is denoted by X^*. For any given multivalued mapping T from a nonempty convex subset K of X into X^* with nonempty values, the generalized variational inequality GVI$(T,\ K)$ is the problem to find $(x,\ y) \in \mathcal{G}(T)$ such that

$$\langle y,\ u - x\rangle \geq 0 \quad \text{for all } u \in K.$$

Such a pair $(x,\ y)$ will be called a solution of the problem GVI$(T,\ K)$. When every $T(x)$ is a singleton, the problem GVI$(T,\ K)$ reduces to the variational inequality VI$(T,\ K)$ associated with the single valued operator $T : K \longrightarrow X^*$.

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In literature, most of existence results for generalized variational inequalities are established by requiring the associated mappings to be monotone or algebraic pseudomonotone in the sense of Karamardian. In this paper, we shall derive existence results for generalized variational inequalities associated with multivalued mappings satisfying the $(S)_+$ condition. The $(S)_+$ condition can be regarded as a generalized monotonicity. There is another such a condition, called the $(S)_+^1$ condition, which is closely related to the $(S)_+$ condition. See [6,10] for a discussion on generalized variational inequalities associated with mappings satisfying the $(S)_+^1$ condition.

The $(S)_+$ condition for multivalued operators will be given in Sect. 2. As proved in [13], it turns out that a multivalued operator must satisfy the $(S)_+^1$ condition when it satisfies the $(S)_+$ condition. However, the opposite implication is in general undetermined. In Sect. 2, we shall also give some conditions to ensure that a multivalued operator satisfies the $(S)_+$ condition when it satisfies the $(S)_+^1$ condition.

In Sect. 3, some existence results for generalized variational inequalities will be established. With the arguments given in Sect. 2, we prove an existence result in Theorem 3.3 which generalizes Theorem 4.3 of [6] to any normed space. As an application, in Sect. 4, we consider multivalued complementarity problems associated with mappings satisfying the $(S)_+$ condition, and prove a theorem to characterize the solvability of such problems in terms of exceptional families of elements.

2 The $(S)_+$ condition

Before describing the $(S)_+$ condition, we shall set up some notations that we need in the sequel. Let X be a topological vector space. For given $x \in X$ and $y \in X^*$, we shall use $\langle y, x\rangle$ for the value of y at x. For a net $\{x_\alpha\}$ in X, we write $x_\alpha \longrightarrow x \in X$ when $\{x_\alpha\}$ converges to x in the original topology on X, and write $x_\alpha \xrightarrow{w} x$ when $\{x_\alpha\}$ weakly converges to x.

Let X_s^* denote the space X^* equipped with the weak-star topology, and write $y_\alpha \xrightarrow{w^*} y$ when $\{y_\alpha\}$ is a net in X^* weak-star convergent to $y \in X^*$. There is a strong topology on X^*, called the topology of bounded convergence, which coincides with the norm topology on X^* when X is a normed space. Let X_b^* denote the space X^* equipped with the topology of bounded convergence. Note that X_s^* and X_b^* are Hausdorff topological vector spaces [17, pp. 79–80], and that X_b^* is a Banach space when X is a normed space [17, p. 42].

We are now ready to describe the $(S)_+$ condition, and recall first the $(S)_+$ condition for single valued operators. In this paper, we shall only consider operators defined on nonempty subsets of normed spaces. See [8] and references there in for a discussion on general topological vector spaces or the vectorial $(S)_+$ condition for single valued operators.

Single valued operators

The $(S)_+$ condition for single valued operators defined on nonempty subsets of Banach spaces was introduced by Browder [5]. This condition extends naturally to normed spaces. Let K be a nonempty subset of a normed space X. A single valued operator $T : K \longrightarrow X^*$ is said to satisfy the $(S)_+$ condition if $x_n \longrightarrow x$ for any sequence $\{x_n\}_{n=1}^\infty$ in K satisfying

$$x_n \xrightarrow{w} x \in K \quad \text{and} \quad \liminf_{n\to\infty}\langle T(x_n), x - x_n\rangle \geq 0. \tag{1}$$

There is a *weak* $(S)_+$ condition introduced by Chiang [7] stated as follows. The mapping T given above satisfies the weak $(S)_+$ condition if every sequence $\{x_n\}_{n=1}^{\infty}$ in K with the properties given in (1) has a subsequence converging to x in norm.

For dealing with variational inequalities, the weak $(S)_+$ condition works equally well as the $(S)_+$ condition does. Therefore, from now on, the weak $(S)_+$ condition is referred to as the $(S)_+$ condition.

Multivalued operators

The $(S)_+$ condition for multivalued mappings is formulated to include the (weak) $(S)_+$ condition for single valued operators as a special case. A multivalued mapping T from a nonempty subset K of a normed space X into X^* is said to satisfy the $(S)_+$ condition if for any sequence $\{(x_n,\ y_n)\}_{n=1}^{\infty}$ in $\mathcal{G}(T)$ with

$$x_n \xrightarrow{w} x \in K \quad \text{and} \quad \liminf_{n\to\infty} \langle y_n,\ x - x_n\rangle \geq 0\,, \tag{2}$$

there is a subsequence of $\{x_n\}_{n=1}^{\infty}$ converging to x in norm.

In the rest of this section, we shall compare the $(S)_+$ condition to the $(S)_+^1$ condition. The $(S)_+^1$ condition for single valued operators was introduced by Isac and Gowda [13], and that for multivalued mappings was introduced by Cubiotti and Yao [10], stated as follows. The multivalued mapping T given above is said to satisfy the $(S)_+^1$ condition if for any sequence $\{(x_n,\ y_n)\}_{n=1}^{\infty}$ in $\mathcal{G}(T)$ with

$$x_n \xrightarrow{w} x \in K, \quad y_n \xrightarrow{w^*} y \in X^* \quad \text{and} \quad \liminf_{n\to\infty} \langle y_n,\ x - x_n\rangle \geq 0\,,$$

there is a subsequence of $\{x_n\}_{n=1}^{\infty}$ converging to x in norm. A single valued mapping $T : K \longrightarrow X^*$ satisfies the $(S)_+^1$ condition if the multivalued mapping $x \longmapsto \{T(x)\}$ does. The vectorial $(S)_+^1$ condition for generalized vector variational inequalities has been considered by Chiang in [6].

It follows immediately from the definition that a multivalued mapping T as given above must satisfy the $(S)_+^1$ condition when it satisfies the $(S)_+$ condition. The opposite implication holds trivially whenever $\{y_n\}_{n=1}^{\infty}$ has a weak-star convergent subsequence for any given sequence $\{(x_n,\ y_n)\}_{n=1}^{\infty}$ in $\mathcal{G}(T)$ satisfying the conditions given in (2). Since in a normed space the weak compactness coincides with the sequentially weak compactness [1, Eberlein–Šmulian Theorem, p. 256], we obtain the following theorem.

Theorem 2.1 *Let T be a multivalued mapping from a nonempty subset K of a normed space X into X^* such that $T(K)$ is contained in a weakly compact subset of X_b^*. Then T satisfies the $(S)_+$ condition if and only if T satisfies the $(S)_+^1$ condition.*

To weaken the relatively weak compactness of $T(K)$ in Theorem 2.1, we consider normed spaces X having the property that $\{y \in X_b^* : \|y\| \leq 1\}$ is weak-star sequentially compact. This property will be called the weak-star Bolzano–Weierstrass property. For instance, reflexive Banach spaces and separable normed spaces [14, Sect. 7.2 Theorem 6] have the weak-star Bolzano–Weierstrass property.

In a normed space X having the weak-star Bolzano–Weierstrass property, if a multivalued mapping $T : K \longrightarrow 2^{X_b^*}$ is bounded, then $\{y_n\}_{n=1}^{\infty}$ has a weak-star convergent subsequece whenever $\{(x_n,\ y_n)\}_{n=1}^{\infty}$ is a sequence in $\mathcal{G}(T)$ with $\{x_n\}_{n=1}^{\infty}$ weakly convergent to a point in K. Recall that $T : K \longrightarrow 2^{X_b^*}$ is bounded if it maps every bounded subset of K into a bounded set in X_b^*.

Theorem 2.2 *Let X be a normed space having the weak-star Bolzano–Weierstrass property. If T is a bounded multivalued mapping from a nonempty subset K of X into X_b^*, then T satisfies the $(S)_+$ condition if and only if T satisfies the $(S)_+^1$ condition.*

For barreled normed spaces X, a subset of X^* is bounded in X_b^* if and only if it is bounded in X_s^*; see Corollary 4.1 and Theorem 4.2 in [17, p. 83]. Thus, we have:

Theorem 2.3 *Let X be a barreled normed space having the weak-star Bolzano–Weierstrass property. If T is bounded multivalued mapping from a nonempty subset K of X into X_s^*, then T satisfies the $(S)_+$ condition if and only if T satisfies the $(S)_+^1$ condition.*

3 Generalized variational inequalities

To derive existence results for generalized variational inequalities, we need some preliminary definitions and results for multivalued mappings. Let $T : \Omega \longrightarrow 2^Y$ be a multivalued mapping, where Ω and Y are Hausdorff topological spaces.

(i) T is called upper semicontinuous at $x_0 \in \Omega$ if for any open subset V of Y containing $T(x_0)$ there is a neighborhood $U \subset \Omega$ of x_0 such that $T(U) \subset V$. While T is simply called upper semicontinuous on Ω if T is upper semicontinuous at every point of Ω. Note that if T is upper semicontinuous on Ω and has compact values, and if Ω is compact, then $T(\Omega)$ is compact; see Theorem VI.1.3 in [3, p. 110].

(ii) T is said to be closed if its graph $\mathcal{G}(T)$ is a closed subset of the product space $\Omega \times Y$. If T is upper semicontinuous on Ω and has closed values, then T is closed [2, Sect. 1.1, Proposition 2, p. 41].

We are now ready to derive existence results, and start with considering multivalued mappings defined on compact and convex subsets of topological vector spaces.

Theorem 3.1 *Let K be a nonempty compact and convex subset of a Hausdorff topological vector space X, and let T be an upper semicontinuous multivalued mapping from K into X_s^* with nonempty convex and compact values.*

(i) *If $T(K)$ is bounded in X_b^*, then* GVI(T, K) *has a solution.*

(ii) *If X is locally convex and barreled, then* GVI(T, K) *has a solution.*

Proof From [16, Theorem 3.2], the assertion (i) follows if for every $u \in K$,

$$\Omega(u) = \{x \in K : \langle y, u - x\rangle \geq 0 \quad \text{for some } y \in T(x)\}$$

is closed. Notice that T is closed. By a similar argument as in the proof of [6, Proposition 2.2], the compactness of $T(K)$ in X_s^* together with its boundedness in X_b^* will imply that every $\Omega(u)$ is closed.

When X is locally convex and barreled, the compactness of $T(K)$ in X_s^* implies the boundedness of $T(K)$ in X_b^*; see [17, Corollary 4.1 and Theorem 4.2, p. 83]. The assertion (ii) now follows from (i) immediately.

Theorem 3.2 *Let K be a nonempty weakly compact and convex subset of a normed space X, and let T be a multivalued mapping from K into X^*. Then* GVI(T, K) *has a solution if the following conditions are satisfied.*

(i) *T has nonempty convex and closed values.*

(ii) $T : co(E) \longrightarrow 2^{X_s^*}$ *is upper semicontinuous for every nonempty compact set* $E \subset K$.
(iii) T *satisfies the* $(S)_+$ *condition.*
(iv) $T(K)$ *is bounded in* X_b^*.

Proof The condition (iv) together with Alaoglu's Theorem [9, p. 134] imply that $T(K)$ is relatively compact in X_s^*. Then, by the condition (ii), T has compact values, and $T(E)$ is compact in X_s^* for every compact subset E of K.

Let $\mathcal{F}$ denote the family of all nonempty finite subsets of K. It follows from Theorem 3.1 that for every $E \in \mathcal{F}$,

$$\Omega(E) = \{x \in K : \text{there exists } y \in T(x) \text{ such that } \langle y,\ u - x\rangle \geq 0 \quad \text{for all } u \in \text{co}(E)\} \neq \emptyset.$$

Let $\overline{\Omega(E)}^w$ denote the weak closure of $\Omega(E)$. Clearly, $\Omega(E_1 \cup E_2) \subset \Omega(E_1) \cap \Omega(E_2)$ for any $E_1, E_2 \in \mathcal{F}$. This implies that $\{\overline{\Omega(E)}^w : E \in \mathcal{F}\}$ has the finite intersection property. Since K is weakly compact, we obtain

$$\Omega = \bigcap_{E \in \mathcal{F}} \overline{\Omega(E)}^w \neq \emptyset.$$

Choose any fixed $\widehat{x} \in \Omega$. For every $E \in \mathcal{F}$, we write $\widehat{E} = E \cup \{\widehat{x}\}$ and

$$A(E) = \{y \in T(\widehat{x}) : \langle y,\ u - \widehat{x}\rangle \geq 0 \quad \text{for all } u \in \text{co}(\widehat{E})\}.$$

We claim that

$$A = \bigcap_{E \in \mathcal{F}} A(E) \neq \emptyset.$$

This claim will complete the proof. Indeed, if $\widehat{y} \in A$, then $\langle \widehat{y},\ (1-t)\widehat{x} + tu - \widehat{x}\rangle \geq 0$ for every $u \in K$ and for $0 \leq t \leq 1$. In particular, $\langle \widehat{y},\ u - \widehat{x}\rangle \geq 0$. This proves that $(\widehat{x},\ \widehat{y})$ is a solution of GVI$(T,\ K)$.

For the proof of the claim, we first note that every $A(E)$ is closed in X_s^*. Indeed, if $\{y_\alpha\}$ is a net in $A(E)$ with $y_\alpha \xrightarrow{w^*} y \in X^*$, then for $u \in \text{co}(\widehat{E})$,

$$\langle y,\ u - \widehat{x}\rangle = \lim_\alpha \langle y_\alpha,\ u - \widehat{x}\rangle \geq 0.$$

The compactness of $T(\widehat{x})$ implies that $y \in T(\widehat{x})$ and $y \in A(E)$.

Note that $A(E_1 \cup E_2) \subset A(E_1) \cap A(E_2)$ for $E_1, E_2 \in \mathcal{F}$. By the compactness of $T(\widehat{x})$, the claim will follow if $A(E) \neq \emptyset$ for every $E \in \mathcal{F}$. Since $\widehat{x} \in \overline{\Omega(\widehat{E})}^w$, there is a sequence $\{x_n\}_{n=1}^\infty$ in $\Omega(\widehat{E})$ such that $x_n \xrightarrow{w} \widehat{x}$ [15, Sect. 24, 1.7, p. 313]. For every n, there exists $y_n \in T(x_n)$ such that $\langle y_n,\ u - x_n\rangle \geq 0$ for all $u \in \text{co}(\widehat{E})$. In particular, $\langle y_n,\ \widehat{x} - x_n\rangle \geq 0$ for every n. The condition (iii) implies that $\{x_n\}_{n=1}^\infty$ has a subsequence $\{x_{p(n)}\}_{n=1}^\infty$ converging to $\widehat{x}$ in norm. This subsequence together with $\widehat{x}$ form a compact subset E_0 of K. By the compactness of $T(E_0)$, $\{y_{p(n)}\}_{n=1}^\infty$ has a subnet $\{y_\alpha\}$ converging in X_s^* to some $y \in X^*$. Since the set $\{(x,\ y) \in \text{co}(E_0) \times X^* : y \in T(x)\}$ is closed in $K \times X_s^*$, and since $(x_\alpha,\ y_\alpha) \longrightarrow (\widehat{x},\ y)$, we have $y \in T(\widehat{x})$. Finally, since $T(K)$ is bounded in X_b^*, it follows that for every $u \in \text{co}(\widehat{E})$,

$$\langle y,\ u - \widehat{x}\rangle = \lim_\alpha \langle y_\alpha,\ u - x_\alpha\rangle \geq 0.$$

Therefore, $y \in A(E)$ and $A(E) \neq \emptyset$.

From Theorems 2.1 and 3.2, we obtain the following result which generalizes Theorem 4.3 of [6] to any normed space.

Theorem 3.3 *Let K be a nonempty weakly compact and convex subset of a normed space X, and let T be a multivalued mapping from K into X^* satisfying the conditions (i) and (ii) in Theorem 3.2. If T satisfies the $(S)^1_+$ condition and $T(K)$ is contained in a weakly compact subset of X^*_b, then* $\mathrm{GVI}(T,\ K)$ *has a solution.*

When X is a normed space having the weak-star Bolzano–Weierstrass property, from Theorems 2.2. and 3.2 we obtain some more existence results analogous to Theorem 3.3. For barreled normed spaces, we have the following analogue of Theorem 3.2.

Theorem 3.4 *Let K be a nonempty weakly compact and convex subset of a barreled normed space X, and let T be a multivalued mapping from K into X^*. Then* $\mathrm{GVI}(T,\ K)$ *has a solution if the following conditions are satisfied.*

(i) *T has nonempty convex and compact values.*
(ii) *$T : co(E) \longrightarrow 2^{X^*_s}$ is upper semicontinuous for every nonempty compact set $E \subset K$.*
(iii) *T satisfies the $(S)_+$ condition.*

Proof The assertion follows from Theorem 3.2 and its proof whenever $A(E) \neq \emptyset$ for every $E \in \mathcal{F}$, where $A(E)$ and $\mathcal{F}$ are given in the proof of Theorem 3.2.

For every $E \in \mathcal{F}$, let $\Omega(\widehat{E})$ be given as before. The condition (iii) implies that there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \longrightarrow \widehat{x}$. Let $y_n \in T(x_n)$ be such that $\langle y_n,\ u - x_n\rangle \geq 0$ for all $u \in \mathrm{co}(\widehat{E})$. Let E_0 denote the compact subset of K consisting of all x_n together with $\widehat{x}$. By the same reasoning as in the proof of Theorem 3.2, $\{y_n\}_{n=1}^{\infty}$ has a subnet $\{y_\alpha\}$ converging in X^*_s to some $y \in T(\widehat{x})$. Since X is barreled, $T(E_0)$ is bounded in X^*_b. This implies that $\{y_\alpha\}$ is bounded in X^*_b, and that for every $u \in \mathrm{co}(\widehat{E})$,

$$\langle y,\ u - \widehat{x}\rangle = \lim_{\alpha}\langle y_\alpha,\ u - x_\alpha\rangle \geq 0\,.$$

The proof is complete.

As a consequence of Theorems 2.3 and 3.4, we obtain:

Theorem 3.5 *Assume that X is a barreled normed space having the weak-star Bolzano–Weierstrass property. Let K be a nonempty weakly compact and convex subset of X, and let T be a bounded multivalued mapping from K into X^*_s satisfying the conditions (i) and (ii) in Theorem 3.4. If T satisfies the $(S)^1_+$ condition, then* $\mathrm{GVI}(T, K)$ *has a solution.*

For a given single valued mapping $T\ :\ K \longrightarrow X^*$, the condition (ii) in Theorem 3.4 reduces to the continuity of the mapping $T : \mathrm{co}(E) \longrightarrow X^*_s$ for every nonempty compact set $E \subset K$. In this case, we obtain existence results analogous to Theorems 3.4 and 3.5 for $\mathrm{VI}(T,\ K)$.

4 Exceptional families of elements

As applications of existence theorems obtained in Sect. 3, we shall prove existence results for complementarity problems on Hilbert spaces associated with mappings satisfying the $(S)_+$ condition, and characterize the solvability of such complementarity problems via exceptional families of elements.

Throughout this section, let H denote a Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$, and fix once for all a nonempty closed convex cone K in H with the dual cone:

$$K^* = \{y \in H : \langle y,\ x\rangle \geq 0 \quad \text{for all } x \in K\}.$$

For a given multivalued mapping $T : K \longrightarrow 2^H$, the multivalued complementarity problem MCP(T, K) is to find a pair $(\widehat{x}, \widehat{y}) \in \mathcal{G}(T)$ such that $\widehat{y} \in K^*$ and $\langle \widehat{y}, \widehat{x} \rangle = 0$. Such a pair $(\widehat{x}, \widehat{y})$ is called a solution of MCP(T, K). It is well known that MCP(T, K) has a solution if and only if GVI(T, K) does; see e.g. Sect. 2.3 in [12] for a discussion.

For any given single valued mapping $T : K \longrightarrow H$, by considering the multivalued mapping $x \longmapsto \{T(x)\}$, the corresponding multivalued complementarity problem becomes the nonlinear complementarity problem NCP(T, K) of finding $\widehat{x} \in K$ such that $T(\widehat{x}) \in K^*$ and $\langle T(\widehat{x}), \widehat{x} \rangle = 0$. Note that NCP($T$, K) has a solution if and only if VI(T, K) does.

For a multivalued mapping T given above, a family $\{x_r\}_{r>0}$ of elements of K is called an exceptional family of elements for (T, K) [11, Sect. 8.3], denoted by EFE(T, K), if $\lim_{r\to\infty} \|x_r\| = +\infty$, and for every $r > 0$ there is a real number $\mu_r > 0$ and an $y_r \in T(x_r)$ such that

$$\mu_r x_r + y_r \in K^* \quad \text{and} \quad \langle \mu_r x_r + y_r, x_r \rangle = 0\,.$$

When T is single valued, a family $\{x_r\}_{r>0} \subset K$ is called an EFE(T, K) if the above conditions hold with y_r replaced by $T(x_r)$ for every $r > 0$.

Theorem 4.1 *Let T be a multivalued mapping from K into H with nonempty convex and compact values. If*

(i) *T satisfies the $(S)_+$ condition, and*
(ii) *T is upper semicontinuous on co(E) for every nonempty compact set $E \subset K$.*

Then either MCP(T, K) *has a solution or there is an* EFE(T, K).

Proof We assume that MCP(T, K) has no solutions, and prove that an EFE(T, K) exists. For every $r > 0$, the set $K_r = \{x \in K : \|x\| \leq r\}$ is weakly compact. It follows from Theorem 3.4 that there exist $x_r \in K_r$ and $y_r \in T(x_r)$ such that $\langle y_r, u - x_r \rangle \geq 0$ for all $u \in K_r$. By assumption, (x_r, y_r) is not a solution of GVI(T, K) for every $r > 0$. From Theorem 5.1 of [4], we conclude that $\{x_r\}_{r>0}$ is an EFE(T, K).

Remark 4.2 Since H is barreled and has the weak-star Bolzano–Weierstrass property, the condition (i) in Theorem 4.1 can be replaced by requiring that T is bounded and satisfies the $(S)^1_+$ condition; see Theorem 3.5.

Theorem 4.3 *Let $T : K \longrightarrow H$ be a mapping continuous on co(E) for every nonempty compact set $E \subset K$. If either*

(i) *T satisfies the $(S)_+$ condition, or*
(ii) *T is bounded and satisfies the $(S)^1_+$ condition,*

then either NCP(T, K) *has a solution or there is an* EFE(T, K).

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